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Analytic continuation of hypergeometric functions by complex single loop Euler transforms

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Abstract. The analytic continuation of ${}_2F_1(a, b; 1; z)$ into the region $|z| > 1$ is derived by using a complex single loop Euler transform instead of the usual Mellin–Barnes transform. This facilitates a general application of Nordsieck's technique, for the evaluation of space integrals over a product of Coulomb wavefunctions, in which *single*-loop Euler transforms of the ${}_1F_1$ functions lead to considerable topological simplification. Due attention is given to the necessary phase specifications. The method is also contrasted with the single-loop Euler-transform derivation of the asymptotic expansion of ${}_1F_1(b; 1; z)$ for $|z| \gg 1$, which follows Mott and Massey.

1. Analytic continuation of ${}_2F_1$

In one version of the Vainshtein theory for electron–atom collisions (Crothers 1967) we required the following specific contour integral:

$$J(z) \equiv \frac{1}{2\pi \mathrm{i}} \oint^{(0+,1+)} \left(\frac{t}{t-1}\right)^b (1 - tz)^{-a} \frac{\mathrm{d}t}{t} \tag{1}$$

for $|z| > 1$ and a, b nonintegral. The principal branch of $(1-tz)^{-a}$ is assumed, while $\arg(t) = \arg(t-1) = 0$ for any real $t \in (1, +\infty)$ defines the branch of $\{t/(t-1)\}^b$. The cuts are shown in figure 1, the contour excluding z^{-1}. For $|z| > 1, \arg(z) = 0$ is precluded by the prevailing geometry. For $|z| < 1$, the contour can always be sufficiently confined so that for all t on the contour, $|tz| < 1$ and $(1-tz)^{-a}$ can be expanded in the usual binomial series ${}_1F_0(a; \ ; z)$. Then uniform convergence permits term-by-term integration, using

$$\frac{1}{2\pi \mathrm{i}} \oint^{(0+,1+)} \left(\frac{t}{t-1}\right)^b t^{n-1}\, \mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \oint^{(0+)} \frac{(1-s)^{-b}}{s^{n+1}}\, \mathrm{d}s \tag{2}$$

$$= \frac{(b)_n}{n!} \tag{3}$$

where $s = t^{-1}$ and $(b)_n$ is the Pochhammer symbol. Thus we have rapidly obtained the standard result that for $|z| < 1$, $J(z)$ is just the Euler integral representation of the Gauss hypergeometric function ${}_2F_1(a, b; 1; z)$. This is not surprising in that $J(z)$ results from the space integral of a plane wave and two confluent hypergeometric functions, both of which are represented by *single*-loop Euler transforms, which facilitates considerable

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topological simplification. The method is due to Nordsieck (1954). Now the analytic continuation of ${}_2F_1(a, b; 1; z)$ to $|z| > 1$ is well known (cf Erdelyi *et al* 1953), although it is often quoted in physics text books (cf Morse and Feshbach 1953 and Landau and Lifshitz 1958) in terms of some unspecified branches of $(-z)^{-a}$ and $(-z)^{-b}$. It is given by the following convergent expression:

$$(1+P(a,b))\exp(\pm a\pi \mathrm{i})z^{-a}\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(1-a)}{}_2F_1\left(a, a; 1-b+a; \frac{1}{z}\right) \tag{4}$$

where the + or − is taken according as $\arg(z) \in (0, \pi]$ or $[-\pi, 0)$, the principal branches of z^{-a} and z^{-b} are taken and $P(a, b)$ is the permutation operator interchanging a and b. According to Kampé de Fériet (1937), Goursat (1881) was the first to establish (4) rigorously, Gauss and Kummer both having given the formula not without ambiguity.

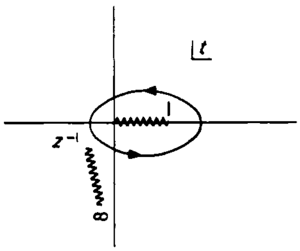

Figure 1. Contour for single-loop Euler transform representation of ${}_2F_1(a, b; 1; z)$.

A detailed proof using Mellin–Barnes transforms is given by Macrobert (1962), following Barnes (1908). It does follow immediately by the theory of analytic continuation that for $|z| > 1$, $J(z)$ is indeed given by expression (4), which is the result obtained by Crothers (1966, 1967) in contrast to the result of Omidvar (1967). This result has also been obtained, for instance by Macrobert (1962, Appendix 2), using double-loop Euler transforms, which are however inappropriate to the Nordsieck method. We shall therefore derive the result (4) directly and rigorously from the single-loop representation given by (1). Such a method seems desirable, since, apart from elegance, there might arise the practical problem of evaluating integrals more complicated than $J(z)$ but still containing $(1-tz)^{-a}$. Coleman (1969), following Crothers (1967), has used conformal mappings to derive (4) directly from (1) in an heuristic manner, obscuring the difficulties concerned with contributions from different sections of the deformed branch cuts and arising from the *apparent* asymmetry of the Euler transform in a and b. These difficulties are overcome by the following simple approach. The branch cut of figure 1 along [0, 1] may be deformed to pass through z^{-1}, as indicated in figure 2, provided there is no change in the relative topology of the cuts and the contour. This requires that the lower part of the contour be pinched at z^{-1} between the two cuts, a legitimate process provided $\mathrm{Re}\, a < 0$. For convenience of evaluation the upper part of the contour may be constrained to touch the cut, joining 0 to 1, at z^{-1}, so that everywhere on the section joining z^{-1} to 1 we have $|tz| > 1$. Then

$$J(z) = J_1 + J_2 \tag{5}$$

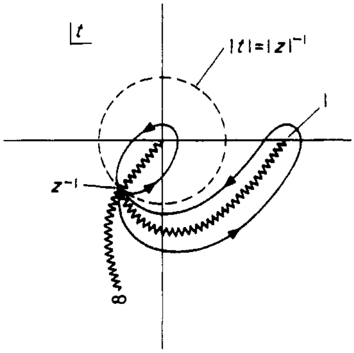

Figure 2. Contour of figure 1 but with cut and contour deformed to facilitate analytic continuation of ${}_2F_1(a, b; 1; z)$ to $|z| > 1$.

where

$$J_1 = \frac{1}{2\pi i}\oint_{1/z}^{(0+)} \left(\frac{t}{t-1}\right)^b (1-tz)^{-a}\frac{dt}{t} \tag{6}$$

$$= \frac{\exp(\pi i b) - \exp(-\pi i b)}{2\pi i}\int_0^{1/z} t^{b-1}(1-t)^{-b}(1-tz)^{-a}\,dt \tag{7}$$

$$= \frac{\sin(\pi b)}{\pi}\sum_{m=0}^{\infty}\frac{(b)_m}{m!}\int_0^{1/z} t^{m+b-1}(1-tz)^{-a}\,dt \tag{8}$$

$$= \frac{\sin(\pi b)}{\pi}\sum_{m=0}^{\infty}\frac{(b)_m}{m!}z^{-m-b}B(b+m, 1-a) \tag{9}$$

$$= \frac{\sin(\pi b)}{\pi}z^{-b}\frac{\Gamma(b)\Gamma(1-a)}{\Gamma(b+1-a)}\,{}_2F_1\left(b, b; b+1-a; \frac{1}{z}\right) \tag{10}$$

and

$$J_2 = \frac{1}{2\pi i}\oint_{1/z}^{(1+)} \left(\frac{t}{t-1}\right)^b (1-tz)^{-a}\frac{dt}{t} \tag{11}$$

$$= \frac{\sin(\pi b)}{\pi}\int_{1/z}^{1} t^{b-1}(1-t)^{-b}(1-tz)^{-a}\,dt \tag{12}$$

$$= \frac{\sin(\pi b)}{\pi}\sum_{m=0}^{\infty}\frac{(b)_m}{m!}\int_{1/z}^{1} t^{m+b-1}(1-tz)^{-a}\,dt \tag{13}$$

$$= \frac{\sin(\pi b)}{\pi}\exp(\pm a\pi i)z^{-a}\sum_{m=0}^{\infty}\frac{(b)_m}{m!}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^{-n}\frac{1-z^{a+n-b-m}}{b-a-n+m} \tag{14}$$

$$= \frac{\sin(\pi b)}{\pi}\Bigg(\exp(\pm a\pi i)z^{-a}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}B(b-a-n, 1-b)z^{-n}$$

$$+\exp(\pm a\pi i)z^{-b}\sum_{m=0}^{\infty}\frac{(b)_m}{m!}B(a-b-m, 1-a)z^{-m}\Bigg). \tag{15}$$

$B(\alpha, \beta)$ is the beta function, $\arg(z)$ is restricted to either $(0, \pi]$ or $[-\pi, 0)$ and the upper or lower sign taken respectively; in equations (7) and (12) the principal branches of $(1-t)^{-b}$

and t^b are taken. In evaluating (7) we have assumed $\operatorname{Re} a < 0, \operatorname{Re} b > 1$ to ensure uniform convergence for term-by-term integration. However the condition on b for J_1 may finally be relaxed by analytic continuation. Similarly in evaluating (12) we have assumed $\operatorname{Re} a < 0, \operatorname{Re} b < 0$; again the condition on b for J_2 may be relaxed. We have also assumed that $(b-a)$ is not a nonnegative integer, otherwise (14) is invalid and the usual logarithmic term arises. Using

$$\Gamma(c-m) = \frac{\pi}{\sin(\pi c)} \frac{(-1)^m}{\Gamma(m-c+1)} \tag{16}$$

we obtain

$$J_2 = \frac{\sin(\pi b)}{\pi} \exp(\pm a\pi \mathrm{i})(1+P(a,b)) \frac{\Gamma(1-b)\Gamma(b-a)}{\Gamma(1-a)} \times z^{-a}{}_2F_1\left(a, a; 1+a-b; \frac{1}{z}\right) \tag{17}$$

and since a, b and $(a-b)$ are nonintegral using (16) with $m = 0$, the following identity may be verified:

$$\frac{\sin(\pi b)}{\pi}\left(\frac{\Gamma(b)\Gamma(1-a)}{\Gamma(b+1-a)} + \exp(\pm a\pi \mathrm{i})\frac{\Gamma(1-a)\Gamma(a-b)}{\Gamma(1-b)}\right) = \frac{\exp(\pm b\pi \mathrm{i})\Gamma(a-b)}{\Gamma(a)\Gamma(1-b)} \tag{18}$$

whereupon (10) and (17) yield the result (4), the condition $\operatorname{Re} a < 0$ on which may be relaxed by analytic continuation.

2. Asymptotic behaviour of ${}_1F_1$

It is interesting to consider the special case

$$\lim_{a \to \infty} J\left(\frac{z}{a}\right) = {}_1F_1(b; 1; z) \tag{19}$$

which is of course convergent for all finite z. The analogue of (4) is the following *asymptotic* expansion:

$$\frac{\mathrm{e}^z z^{b-1}}{\Gamma(b)} {}_2F_0\left(1-b, 1-b; ; \frac{1}{z}\right) + \frac{\exp(\pm b\pi \mathrm{i}) z^{-b}}{\Gamma(1-b)} {}_2F_0\left(b, b; ; -\frac{1}{z}\right) \tag{20}$$

in which the same conventions hold as for expression (4). Evidently if $\arg(z) = 0$ and $|z| \gg 1$ the first term is exponentially dominant while the second term is subdominant. The usual Stokes phenomenon is thus clearly exhibited, the subdominant term changing discontinuously by a multiplicative factor of $\exp(2\pi b\mathrm{i})$ as the Stokes line is crossed in the positive sense. For $\arg(z) = \pm\pi$ and $|z| \gg 1$ the second term is dominant and the first subdominant, changing discontinuously by a factor of $\exp(-2\pi b\mathrm{i})$ as the Stokes line is crossed in the positive sense. It seems worthwhile stating these simple facts, since they are often omitted (cf Erdelyi *et al* 1953, p 278). The usual Coulomb wavefunctions of scattering theory depend on the use of (20) on the anti-Stokes lines $\arg(z) = \pm\pi/2$, upon which neither term is *exponentially* dominating for $|z| \gg 1$ and both are single valued.

Now the second term of (20) is readily obtained from that of (4), but the first term is not so readily deduced. Of course, the result is well known but it is more satisfactory within the present context to obtain it directly from the limit of (1), namely the Euler transform

$$\frac{1}{2\pi i}\oint^{(0+,1+)}\left(\frac{t}{t-1}\right)^{b}\exp(zt)\frac{dt}{t} \tag{21}$$

which gives (19) upon expansion of exp(zt) and application of (3). In fact the required method is given by Mott and Massey (1965) but without the necessary phase specifications. Putting $v = zt$, we then obtain

$${}_1F_1(b;1;z) = \frac{1}{2\pi i}\oint^{(0+,z+)}\left(1-\frac{z}{v}\right)^{-b} e^{v}\frac{dv}{v} \equiv W_1 + W_2 \tag{22}$$

where

$$W_1 = \frac{1}{2\pi i}\oint_{-\infty}^{(0+)}\left(1-\frac{z}{v}\right)^{-b} e^{v}\frac{dv}{v} \tag{23}$$

$$= \exp(\pm b\pi i)z^{-b}\frac{1}{2\pi i}\oint_{-\infty}^{(0+)}\left(1-\frac{v}{z}\right)^{-b} e^{v}v^{b-1}\,dv \tag{24}$$

$$\sim \exp(\pm b\pi i)z^{-b}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}z^{-n}\frac{1}{2\pi i}\oint_{-\infty}^{(0+)} e^{v}v^{n+b-1}\,dv \tag{25}$$

$$= \exp(\pm b\pi i)z^{-b}\sum_{n=0}^{\infty}\frac{(b)_n(b)_n z^{-n}\sin(\pi b)\Gamma(b)}{n!\pi} \tag{26}$$

$$= \frac{\exp(\pm b\pi i)z^{-b}}{\Gamma(1-b)}\,{}_2F_0\left(b,b;\,;\,-\frac{1}{z}\right) \tag{27}$$

and

$$W_2 = \frac{1}{2\pi i}\oint_{-\infty+i\,\mathrm{Im}\,z}^{(z+)}\left(1-\frac{z}{v}\right)^{-b} e^{v}\frac{dv}{v} \tag{28}$$

$$= \frac{e^{z}}{2\pi i}\oint_{-\infty}^{(0+)}\left(1+\frac{z}{u}\right)^{b-1} e^{u}\frac{du}{u} \tag{29}$$

$$\sim \frac{e^{z}z^{b-1}}{\Gamma(b)}\,{}_2F_0\left(1-b,1-b;\,;\,\frac{1}{z}\right). \tag{30}$$

The deformed contour used to evaluate (22) is shown in figure 3. The contributions from BC and DA, taken in the positive sense, are W_1 and W_2 respectively. The contributions from CD and AB are nil, since $\exp(-\infty)$ is zero. Expression (25) is clearly only asymptotic since $|v/z| > 1$ holds for $|v| > |z|$, thus preventing convergence of the binomial series on sections of the contour. The contour integral in (25) is just the Hankel representation of the reciprocal of the gamma function, given that v^b is the principal branch. $\mathrm{Arg}(-z)$ is restricted to $(-\pi, +\pi)$, because we took

$$\left(1-\frac{z}{v}\right) = -z\times v^{-1}\times\left(1-\frac{v}{z}\right) \tag{31}$$

and since $\arg(1-z/v)\in(-\pi,+\pi)$ *a priori*, while $\arg(1-v/z)$ and $\arg(v)\in(-\pi,+\pi)$ in order to justify (25) and (26). Thus in (24) we intend the principal branch of z^{-b} and the upper or lower sign of $\exp(\pm b\pi i)$ according as $\arg(z) \in (0, \pi]$ or $[-\pi, 0)$ respectively. Contour integral (29) is obtained from (28) by the mapping $v = u+z$. Expression (30) follows from (29) as (27) does from (24), so that once again the principal branch of z^b is intended. Thus finally the result (20) follows from (22), (27) and (30). It is also clear that expression (4) cannot be derived by this latter method, ultimately because $z = \infty$ is a regular singular point of ${}_2F_1$ and an essential singularity of ${}_1F_1$.

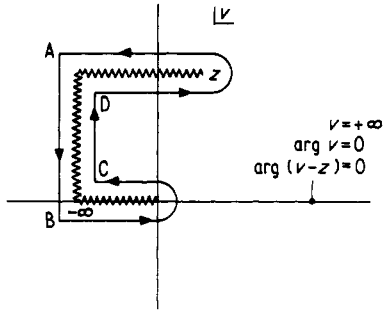

Figure 3. Contour for single-loop Euler transform representation of ${}_1F_1(b; 1; z)$, but with cut and contour deformed to facilitate asymptotic expansion for $|z| \gg 1$.

Evidently the simple methods used here to evaluate expressions (1) and (21) for all z, may be extended to deal with more complicated integrands. No other simple extensions spring to mind which can not be obtained by parametric differentiation. However n fold complex single-loop Euler transforms would facilitate a unified treatment of Appell's hypergeometric function F_2 and other higher generalized hypergeometric functions, in contrast with real transforms (Olsson 1967) which necessitate piece-meal analytic continuation obtained by other techniques.

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